

# Modeling and controlling an active constrained layered (ACL) beam actuated by two voltage sources with/without magnetic effects

Ahmet Özkan Özer\*

\*Department of Mathematics and Statistics, University of Nevada, Reno, Nevada 89557, USA

**Abstract**—A fully dynamic three-layer active constrained layer (ACL) beam model, consisting of two piezoelectric elastic layers constraining a viscoelastic layer, is modeled for clamped-free boundary conditions by using a thorough variational approach. The Rao-Nakra thin complaint layer assumptions are adopted to model the sandwich structure, and all magnetic effects for the piezoelectric layers are retained. The piezoelectric layers are activated by two different voltage sources. When there are no “mechanical” boundary forces acting in the longitudinal direction, it is shown that the system with certain parameter combinations can not be uniformly strongly stabilizable by  $B^*$ -type feedback that are all electrical for the piezoelectric layers. However, as we ignore all magnetic effects (electrostatic assumption), the closed-loop system with all mechanical feedback is shown to be uniformly exponentially stable.

**Index Terms**—ACL beam; Smart sandwich beam, Piezoelectric beam, three-layer Rao-Nakra beam, Voltage controller, Boundary feedback stabilization.

## I. INTRODUCTION

An active constrained layer (ACL) beam consists of two piezoelectric layers and a constrained viscoelastic layer. Each piezoelectric layer is actuated by different voltage sources. When the electrodes of the piezoelectric layers are subjected to voltage sources, they shrink or extend, and therefore, the whole ACL composite shrinks/extends or bends. Accurately modeling the composite requires certain mechanical and electrical (and magnetic) assumptions for each layer. The middle layer is modeled by classical Mindlin-Timoshenko assumptions and the stiff piezoelectric layers are modeled by the Euler-bernoulli assumptions. Piezoelectric layers are traditionally modeled through the electrostatic assumption, and all dynamic electrical effects and magnetic effects due to Maxwell’s equations are ruled out, i.e. see [16], and the references therein. Since an ACL beam includes a piezoelectric layer, the corresponding models use the electrostatic assumption as well [19]. The reduced model ([1], [7], [15]) is mostly either a Mead-Marcus-type [8] or a Rao-Nakra-type model [13]. For example, [1] obtained a Mead-Marcus type model by neglecting the rotational inertia terms for the longitudinal dynamics and rotational inertia for the bending dynamics.

All of these models reduce to the classical counterparts as in ([3], [5], [8]) once the piezoelectric strain is taken to be zero. On the other hand, the model obtained in [15] through a consistent variational approach is more like a Rao-Nakra-type [13].

As the electrostatic assumption is adopted, it can be easily shown that a single piezoelectric beam model is exactly controllable, and uniformly exponentially stabilizable for the  $B^*$ -type mechanical feedback, velocity of the beam at one end. The same type of phenomenon is observed for the ACL beam. For example in [1], it is shown that the time derivative of the energy is nonnegative as a mechanical damping is applied through the boundary of the piezoelectric layer. See other control strategies i.e. in [17], and the references therein. However, even in the case of a single piezoelectric beam, as the dynamic effects are kept, a strongly coupled wave system is obtained for which it is shown that the model is not controllable and not uniformly exponentially stabilizable for almost all combinations of material parameters with the  $B^*$ -type feedback, i.e. the total current at the electrodes [10]. In fact, there are no feedback controllers to make system uniformly exponentially stable. It is worthwhile to mention that the closed-loop system obtained by supplying voltage to the electrodes and feeding back the total current is much easier and physical in terms of practical applications since measuring the total current at the electrodes is way easier than measuring displacements or the velocity of the composite at one end of the beam, i.e. see ([2], [9]). In fact, with the same feedback controller, explicit polynomial decay estimates are obtained for more regular initial data [11]. These results for a single piezoelectric beam indicate that the closed-loop system for an ACL beam with a similar type of input-output mentioned above may be more physical, and moreover, the ACL beam models with the magnetic effects may have the same type of stabilizability/controllability characteristics of a single piezoelectric beam.

In this paper, a novel modeling strategy is proposed to obtain a reduced (but fully dynamic model) an ACL beam with/without the magnetic effects by using a modified Lagrangian by ignoring the weight and the stiffness of the middle layer, i.e. see [4]. In fact, more attention is paid only to the piezoelectric layers in the modeling process by

using the full set of Maxwell's equations. The obtained model differs from the classical counterparts substantially. This model without the mechanical boundary forces for the longitudinal dynamics is shown to be not uniformly strongly stable with the  $B^*$ -type feedback, the total current flowing through the electrodes of the piezoelectric layers. Once the magnetic effects are ignored (electrostatic model), our model simplifies to the reduced Rao-Nakra beam model [4] with clamped-free boundary conditions. To our knowledge, the boundary feedback stabilization or exact controllability for the clamped-free Rao-Nakra beam model were never considered in the literature, i.e. see [12] and the references therein. It is shown that the reduced model is a compact perturbation of the decoupled system consisting of a decoupled system: an exponentially stable Rayleigh beam equation [14] and two wave equations [12]. Therefore the coupled system is shown to be exponentially stable since the unique continuation result proves that there are no eigenvalues on the imaginary axis, see Lemma 4.2.

## II. MODELING ACL BEAMS

Consider an ACL beam occupying the region  $\Omega = \Omega_{xy} \times (0, h) = [0, L] \times [-b, b] \times (0, h)$  at equilibrium where  $\Omega_{xy}$  is a smooth bounded domain in the plane. The total thickness  $h$  is assumed to be small in comparison to the dimensions of  $\Omega_{xy}$ . The beam consists of two piezoelectric layers and a compliant layer. The layers are indexed from 1 to 3 from the bottom piezoelectric layer to the top piezoelectric layer, respectively. Now we let  $0 = z_0 < z_1 < z_2 < z_3 = h$ , with  $h_i = z_i - z_{i-1}$ ,  $i = 1, 2, 3$ . We use the rectangular coordinates  $(x, y)$  to denote points in  $\Omega_{xy}$ , and  $(X, z)$  to denote points in  $\Omega = \Omega_B \cup \Omega^{ve} \cup \Omega^T$ , where  $\Omega_B = \Omega_{xy} \times (z_0, z_1)$ ,  $\Omega^{ve} = \Omega_{xy} \times (z_1, z_2)$ , and  $\Omega^T = \Omega_{xy} \times (z_2, z_3)$  are the reference configurations of the bottom piezoelectric, viscoelastic, and top piezoelectric layers, respectively.

For  $(x, y, z) \in \Omega$ , let  $U(x, y, z) = (U_1, U_2, U_3)(x, y, z)$  denote the displacement vector (from reference configuration). In order to obtain a beam theory, all displacements are assumed to be independent of  $y$ -coordinate, and  $U_2(x) = 0$  for all  $x \in \Omega$ . The transverse displacements is  $w(x) = w^i(x)$  for any  $i$  and  $x \in [0, L]$ . Define  $u^i(x)$  for  $i = 1, 2, 3$  by  $u^i = U_i(x, 0, z_i)$ , for  $i = 1, 2, 3$  for all  $x \in (0, L)$ .

We use the standard sandwich beam assumptions to model the ACL beams. The modified constitutive equations for the piezoelectric layers are

$$\begin{cases} T_{11}^i = \alpha^i S_{11}^{(i)} - \gamma^i \beta^i D_3^i, & E_1^i = \beta_1^i D_1^i \\ E_3^i = -\gamma^i \beta^i S_{11}^i + \beta^i D_3^i, & i = 1, 3 \end{cases} \quad (1)$$

where  $\alpha^i = \alpha_1^i + (\gamma^i)^2 \beta^i$ ,  $\alpha_1^i = c_{11}^i$ ,  $\alpha^2 = c_{11}^2$  and  $\gamma^i = \gamma_{31}^i$ ,  $\gamma_1^i = \gamma_{15}^i$ ,  $\beta^i = \frac{1}{\varepsilon_{33}^i}$ ,  $\beta_1^i = \frac{1}{\varepsilon_{11}^i}$ , and the middle layer is  $T_{11} = \alpha_1^2 S_{11}$ ,  $T_{13} = 2G_2 S_{13}$  where  $G_2$  is the shear modulus of the viscoelastic layer, and refer to [10] for the description of piezoelectric and elasticity coefficients. Defining  $\hat{z}^i = \frac{z^{i-1} + z^i}{2}$ , the strain components

for the viscoelastic layer, and the piezoelectric layers are respectively given by

$$\begin{aligned} S_{11} &= \frac{\partial v^2}{\partial x} - (z - \hat{z}_i) \frac{\partial \psi^2}{\partial x}, & S_{13} &= \frac{1}{2} (\psi^2 + w_x) = \frac{1}{2} \phi^2, \\ S_{11}^i &= \frac{\partial v^i}{\partial x} - (z - \hat{z}_i) \frac{\partial^2 w}{\partial x^2}, & S_{13}^i &= 0, \quad i = 1, 3 \\ \psi^i &= \frac{u^i - u^{i-1}}{h_i}, & \phi^i &= \psi^i + w_x, \quad v^i = \frac{u^{i-1} + u^i}{2} \end{aligned} \quad (2)$$

where  $i = 1, 2, 3$ , and in particular,  $\phi^1 = \phi^3 = 0$ ,  $\psi^1 = \psi^3 = -w_x$ ,  $\phi^2 = \psi^2 + w_x$ . Here  $\psi^i$  can be viewed as the total rotation angles of the deformed filament within the  $i^{\text{th}}$  layer in the  $x - z$  plane, and  $\phi^i$  represent the shear angles within each layer, and  $v^i$  represent the longitudinal displacements of the center line of the  $i^{\text{th}}$  layer. For details of the constitutive equations and parameters, the reader may refer to ([10], [4]).

**Inclusion of the electrical kinetic energy for the piezoelectric layers:** We follow the dynamic approach in [10] to use the full set of Maxwell's equations. Let  $B^i$  be the magnetic field  $B^i(x)$  for the  $i^{\text{th}}$  piezoelectric layer for  $i = 1, 3$ , and have the only nonzero component  $B_2^i(x)$ . Assume also that the electric field of the  $i^{\text{th}}$  piezoelectric layer  $E_1^i = 0$ , and thus  $D_1^i = 0$ . Assuming that  $D_3^i$  does not vary in the thickness direction  $D_3^i(x, z, t) = D_3^i(x, t)$ , it follows from the Ampère-Maxwell equation that  $B_2^i = -\mu^i \int_0^x \dot{D}_3^i(\xi, z, t) d\xi$  where  $\mu^i$  is the magnetic permeability of the  $i^{\text{th}}$  layer. Now we define  $p^i = \int_0^x D_3^i(\xi, t) d\xi$  to be the total electric charge at point  $x$ . The magnetic energy for the  $i^{\text{th}}$  layer is  $B^i = \frac{\mu^i}{2} \int_{\Omega} (\dot{p}^i)^2 dX$ .

Assume that the beam is subject to a distribution of boundary forces  $(\tilde{g}^1, \tilde{g}^3, \tilde{g})$  along its edge  $x = L$ . Let  $V^T(t)$  and  $V_B(t)$  be the voltages applied at the electrodes of the piezoelectric layers, respectively. Then the total work done by all mechanical and electrical external forces is

$$\begin{aligned} \mathbf{W} &= \int_0^L \left( -(p^1)_x V_B - (p^3)_x V^T \right) dx + g^1 v^1(L) \\ &\quad + g^3 v^3(L) + gw(L) - Mw_x(L). \end{aligned} \quad (4)$$

The modified Lagrangian for the ACL beam is

$$\mathbf{L} = \int_0^T [\mathbf{K} - (\mathbf{P} + \mathbf{E}) + \mathbf{B} + \mathbf{W}] dt \quad (5)$$

where  $\mathbf{K} = \sum_{i=1}^3 \mathbf{K}^i$ ,  $\mathbf{P} + \mathbf{E} = \mathbf{P}^2 + \sum_{i=1,3} (\mathbf{P}^i + \mathbf{E}^i)$ , and  $\mathbf{B} = \mathbf{B}^1 + \mathbf{B}^3$  are the kinetic energy, the total stored energy, and the magnetic energy of the beam [10],

$$\begin{aligned} \mathbf{K} &= \frac{1}{2} \int_0^L \left[ \left( \sum_{i=1,3} \rho_i h_i (\dot{v}^i)^2 \right) + \left( \sum_{i=1,3} \rho_i h_i \right) \dot{w}^2 \right. \\ &\quad \left. + \rho_2 h_2 (\dot{\psi}^2)^2 + (\rho_1 h_1 + \rho_3 h_3) \dot{w}_x^2 \right] dx, \\ \mathbf{P} + \mathbf{E} &= \frac{1}{2} \int_0^L \left[ \alpha^2 h_2 \left( (v_x^2)^2 + \frac{h_2^2}{12} (\psi_x^2)^2 \right) \right. \\ &\quad \left. + G_2 h_2 (\phi^2)^2 + \sum_{i=1,3} \left( \alpha^i h_i \left( (v_x^i)^2 + \frac{h_i^2}{12} (w_{xx})^2 \right) \right. \right. \\ &\quad \left. \left. - 2\gamma^i \beta^i h_i v_x^i p_x^i + \beta^i h_i (p_x^i)^2 \right) \right] dx, \\ \mathbf{B} &= \frac{1}{2} \int_0^L \sum_{i=1,3} (\mu^i h_i (\dot{p}^i)^2) dx \end{aligned} \quad (6)$$

where  $\rho_i$  is the volume density of the  $i$ -th layer. Refer ([4], [10]) for the details.

### III. RAO-NAKRA MODEL AND HAMILTON'S PRINCIPLE

By using (3),  $\{v^2, \phi^2, \psi^2\}$  can be written as functions of  $\{w, v^1, v^3\}$ . Thus, we choose  $w, v^1, v^3$  as the state variables. Let  $H = \frac{h_1 + 2h_2 + h_3}{2}$ . Application of Hamilton's principle, by using cantilevered boundary conditions and by setting the variation of admissible displacements  $\{v^1, v^3, p^1, p^3, w\}$  of  $\tilde{\mathbf{L}}$  to zero yields a highly coupled equations for bending and stretching of the whole composite. Since it is not very easy to analyze the controllability properties of this strongly coupled system, we will study the thin compliant layer Rao-Nakra model by letting  $\rho_2, \alpha^2 \rightarrow 0$ . This approximation retains the potential energy of shear and transverse kinetic energy so that the model above reduces to

$$\begin{cases} \rho_i h_i \ddot{v}^i - \alpha^i h_i v_{xx}^i + \gamma^i \beta^i h_i p_{xx}^i + \kappa(i) G_2 \phi^2 = 0, \\ \mu^i h_i \ddot{p}^i - \beta^i h_i p_{xx}^i + \gamma^i \beta^i h_i v_{xx}^i = 0, \quad i = 1, 3, \\ m \ddot{w} - K_1 \ddot{w}_{xx} + K_2 w_{xxxx} - G_2 H \phi_x^2 = 0, \\ \phi^2 = \frac{1}{h_2} (-v^1 + v^3 + H w_x) \end{cases} \quad (7)$$

with the boundary and initial conditions

$$\begin{cases} v^i(0) = p^i(0), \\ \alpha^i h_i v_x^i(L) - \gamma^i \beta^i h_i p_x^i(L) = g^i(t), \quad i = 1, 3 \\ \beta^1 h_1 p_x^1(L) - \gamma^1 \beta^1 h_1 v_x^1(L) = -V_B(t) \\ \beta^3 h_3 p_x^3(L) - \gamma^3 \beta^3 h_3 v_x^3(L) = -V^T(t) \\ w(0) = w_x(0) = 0, \quad K_2 w_{xx}(L) = -M(t) \\ K_1 \ddot{w}_x(L) - K_2 w_{xxxx}(L) + G_2 H \phi^2(L) = g(t) \\ (v^1, v^3, p^1, p^3, w, \dot{v}^1, \dot{v}^3, \dot{p}^1, \dot{p}^3, \dot{w})(x, 0) \\ = (v_0^1, v_0^3, p_0^1, p_0^3, w_0, v_1^1, v_1^3, p_1^1, p_1^3, w_1). \end{cases} \quad (8)$$

where  $\kappa(i) = \text{sgn}(i - 2)$ ,  $m = \sum_{i=1}^3 \rho_i h_i$ ,  $K_1 = \frac{\rho_1 h_1^3}{12} + \frac{\rho_3 h_3^3}{12}$ , and  $K_2 = \frac{\alpha^1 h_1^3}{12} + \frac{\alpha^3 h_3^3}{12}$ . Note that, different from a single piezoelectric model, the voltage controls  $V^T(t)$  and  $V_B(t)$  strongly couple the stretching and bending equations due the shear effect  $\phi^2$  of the middle layer.

Note that the role of the mechanical boundary feedback controllers  $g^1$  and  $g^3$  applied to the beam longitudinally are crucial to obtain a uniform exponential stabilization result (one control for each equation). However, once the mechanical boundary controllers for the stretching equations are removed, i.e.  $g^1, g^3 \equiv 0$ , the system is not even strongly stable by the  $B^*$ -type feedback for certain choices of material parameters. For  $i = 1, 3$  define

$$\zeta_{\pm}^i = \frac{\sqrt{\left(\frac{(\gamma^i)^2 \mu^i}{\alpha_1^i} + \frac{\mu^i}{\beta^i} + \frac{\rho_i}{\alpha_1^i} \pm \sqrt{\left(\frac{(\gamma^i)^2 \mu^i}{\alpha_1^i} + \frac{\mu^i}{\beta^i} + \frac{\rho_i}{\alpha_1^i}\right)^2 - \frac{4\rho_i \mu^i}{\beta^2 \alpha_1^i}}\right)}{\sqrt{2}}$$

$$b_{\pm}^i = \frac{1}{2} \left( \gamma^i + \frac{\alpha_1^i}{\gamma^i \beta^i} - \frac{\rho_i}{\gamma^i \mu^i} \pm \sqrt{\left(\gamma^i + \frac{\alpha_1^i}{\gamma^i \beta^i} - \frac{\rho_i}{\gamma^i \mu^i}\right)^2 + \frac{4\rho_i}{\mu^i}} \right)$$

where  $\zeta_+, \zeta_-, b_-, b_+ \neq 0, b_- \neq b_+, \zeta_+ \neq \zeta_-$  with  $\zeta_+ \zeta_- = \sqrt{\frac{\rho_i \mu^i}{\beta^2 \alpha_1^i}}, b_- b_+ = \frac{\rho_i}{\mu^i}$ . Let two piezoelectric beams have the same material properties, i.e.  $\alpha_1^1 = \alpha_1^3, \gamma^1 = \gamma^3$ , etc.

**Theorem 3.1:** The system (7)-(8) is not strongly stable by the feedback, i.e.  $V^T(t) = \frac{s_3 \dot{p}^3(L)}{2h_3}$ ,  $V_B(t) = \frac{s_1 \dot{p}^1(L)}{2h_1}$ ,  $M(t) = -k_1 \dot{w}_x(L)$  and  $g(t) = k_2 \dot{w}(L)$  for  $s_1, s_2, k_1, k_2 > 0$  if  $\frac{\zeta_{\pm}^i}{\zeta_{\mp}^i} = \frac{2n_i - 1}{2m_i - 1}$  for some  $m_i, n_i \in \mathbb{N}, i = 1, 3$ .

**Proof:** We prove that there are eigenvalues on the imaginary axis. Consider the eigenvalue problem corresponding to (7)-(8) with  $\lambda = i\tau$ :

$$\begin{cases} \alpha^i h_i v_{xx}^i - \gamma^i \beta^i h_i p_{xx}^i + \kappa(i) G_2 \phi^2 = -\tau^2 \rho_i h_i z^i, \\ \beta^i h_i p_{xx}^i - \gamma^i \beta^i h_i v_{xx}^i = -\tau^2 \mu^i h_i p^i, \quad i = 1, 3, \\ -K_2 w_{xxxx} + G_2 H \phi_x^2 = -\tau^2 (m w - K_1 w_{xx}), \\ \phi^2 = \frac{1}{h_2} (-v^1 + v^3 + H w_x) \end{cases} \quad (9)$$

with the overdetermined boundary conditions

$$\begin{aligned} |w = w_x = v^i = p^i|_{x=0} &= |v_x^i = p_x^i = p^i|_{x=L} = 0, \\ w(L) = w_x(L) = w_{xx}(L) = w_{xxx}(L) &= 0, \quad i = 1, 3. \end{aligned} \quad (10)$$

Let  $w(x) \equiv 0$ , and

$$\begin{aligned} v^i(x) &= k_1^i \frac{a_+^i b_+^i \sin(a_-^i x) - a_- b_-^i \sin(a_+^i x)}{a_+^i a_-^i (b_+^i - b_-^i)} \\ &\quad + k_2^i \frac{-a_+^i \sin(a_-^i x) + a_-^i \sin(a_+^i x)}{a_+^i a_-^i (b_+^i - b_-^i)} \\ p^i(x) &= k_1^i \frac{(a_+^i \sin(a_-^i x) - a_-^i \sin(a_+^i x)) b_1 b_2}{a_+^i a_-^i (b_+^i - b_-^i)} \\ &\quad + k_2^i \frac{a_-^i b_+^i \sin(a_+^i x) - a_+^i b_-^i \sin(a_-^i x)}{a_+^i a_-^i (b_+^i - b_-^i)} \end{aligned}$$

where  $a_+^i = \tau \zeta_+^i = \frac{(2n_i - 1)\pi}{2L}$ ,  $a_-^i = \tau \zeta_-^i = \frac{(2m_i - 1)\pi}{2L}$  for some  $m_i, n_i \in \mathbb{N}, i = 1, 3$ , and

$$\begin{aligned} k_1^i &= \frac{a_-^i b_+^i \sin(a_+^i L) - a_+^i b_-^i \sin(a_-^i L)}{a_+^i a_-^i (b_+^i - b_-^i)}, \\ k_2^i &= -\frac{(a_+^i \sin(a_-^i L) - a_-^i \sin(a_+^i L)) b_+^i b_-^i}{a_+^i a_-^i (b_+^i - b_-^i)}. \end{aligned}$$

Here  $v^1 = v^3, p^1 = p^3, w \equiv 0, \phi^2 \equiv 0$ , and  $(v^1, v^3, p^1, p^3, w)$  is the non-trivial solution of eigenvalue problem (9)-(10). This implies that there are eigenvalues on the imaginary axis;  $\left\{ \pm \frac{ia_+^i}{\zeta_+^i}, \pm \frac{ia_-^i}{\zeta_-^i} \right\}$ . The conclusion follows.  $\square$

### IV. STABILIZATION WITHOUT MAGNETIC EFFECTS

First, assume that the magnetic energy for each layer is zero, i.e.  $\mathbf{B}^i = 0$ . Therefore  $\ddot{p}^1 = \ddot{p}^3 \equiv 0$  in (11). The electrostatic model is the well-known Rao-Nakra model in [4]. The boundary stabilization problem is well studied in [12] for the multi-layer beam clamped at the left end and hinged at the right end.

Finally, for  $k_1, k_2, s_1, s_3 > 0$ , analogous to [6] and [14], we consider the following system

$$\begin{cases} \rho_i h_i \ddot{v}^i - \alpha_1^i h_i v_{xx}^i + \kappa(i) G_2 \phi^2 = 0, \quad i = 1, 3, \\ m \ddot{w} - K_1 \ddot{w}_{xx} + K_2 w_{xxxx} - G_2 H \phi_x^2 = 0, \\ \phi^2 = \frac{1}{h_2} (-v^1 + v^3 + H w_x) \end{cases} \quad (11)$$

with the boundary and initial conditions

$$\begin{aligned} v^i(0) &= w(0) = w_x(0) = 0, \\ \alpha_1^i h_i v_x^i(L) &= -s_i \gamma^i \dot{v}^i(L), \quad i = 1, 3, \\ K_2 w_{xx}(L) &= -k_1 \dot{w}_x(L) \\ K_1 \ddot{w}_x(L) - K_2 w_{xxxx}(L) + G_2 H \phi^2(L) &= k_2 \dot{w}(L) \\ (v^1, v^3, w, \dot{v}^1, \dot{v}^3, \dot{w})(x, 0) &= (v_0^1, v_0^3, w_0, v_1^1, v_1^3, w_1). \end{aligned} \quad (12)$$

**Semigroup well-posedness:** Define

$$\begin{aligned} H_L^1(0, L) &= \{\psi \in H^1(0, L) : \psi(0) = 0\}, \\ H_L^2(0, L) &= \{\psi \in H^2(0, L) : \psi(0) = \psi_x(0) = 0\}, \end{aligned}$$

and the complex linear spaces

$$\begin{aligned} \mathbb{X} &= \mathbb{L}^2(0, L), \quad \mathbb{V} = (H_L^1(0, L))^2 \times H_L^2(0, L), \\ \mathbb{H} &= \mathbb{X}^2 \times H_L^1(0, L), \quad \mathcal{H} = \mathbb{V} \times \mathbb{H}. \end{aligned}$$

so that  $\mathbb{V} \subset \mathbb{H} \subset \mathbb{X}^3 \subset \mathbb{H}' \subset \mathbb{V}'$ . The natural energy associated with (11)-(12) is

$$\begin{aligned} E(t) &= \frac{1}{2} \int_0^L \left\{ \sum_{i=1,3} (\rho_i h_i |\dot{v}^i|^2 + \alpha_1^i h_i |v_x^i|^2) + m |\dot{w}|^2 \right. \\ &\quad \left. + K_1 |\dot{w}_x|^2 + K_2 |w_{xx}|^2 + G_2 h_2 |\phi^2|^2 \right\} dx. \end{aligned} \quad (13)$$

This motivates definition of the inner product on  $\mathcal{H}$

$$\begin{aligned} \left\langle \begin{bmatrix} u_1 \\ \vdots \\ u_6 \end{bmatrix}, \begin{bmatrix} v_1 \\ \vdots \\ v_6 \end{bmatrix} \right\rangle_{\mathcal{H}} &= \left\langle \begin{bmatrix} u_4 \\ u_5 \\ u_6 \end{bmatrix}, \begin{bmatrix} v_4 \\ v_5 \\ v_6 \end{bmatrix} \right\rangle_{\mathbb{H}} \\ &\quad + \left\langle \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right\rangle_{\mathbb{V}} \\ &= \int_0^L \left\{ \rho_1 h_1 u_5 \dot{v}_5 + \rho_3 h_3 u_6 \dot{v}_6 + m \dot{u}_8 \dot{v}_8 + K_1 (u_8)_x (\bar{v}_8)_x \right. \\ &\quad \left. + \alpha_1^1 h_1 (u_1)_x (\bar{v}_1)_x + \alpha_1^3 h_3 (u_2)_x (\bar{v}_2)_x + K_2 (u_4)_{xx} (\bar{v}_4)_{xx} \right. \\ &\quad \left. + \frac{G_2}{h_2} (-u_1 + u_2 + H(u_4)_x) (-\bar{v}_1 + \bar{v}_2 + H(\bar{v}_4)_x) \right\} dx. \end{aligned}$$

Obviously,  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  does indeed define an inner product, with the induced energy norm since the term  $\| -u_1 + u_2 + H(u_4)_x \|_{L^2(0,L)}$  is coercive, see [4] for the details.

Let  $\tilde{y} = (v^1, v^3, w)$  be the smooth solution of the system of (11)-(12). Assuming the homogenous problem, all external forces are zero, multiplying the equations in (11) by  $\tilde{y}_1, \tilde{y}_3, \in H_L^1(0, L)$  and  $\tilde{y} \in H_L^2(0, L)$ , respectively, and integrating by parts yields

$$\begin{aligned} \int_0^L (\rho_i h_i \ddot{v}^i \tilde{y}_i + \alpha_1^i h_i v_x^i (\tilde{y}_i)_x + \kappa(i) G_2 \phi^2 \tilde{y}_i) dx \\ + s_i \dot{v}^i(L) \tilde{y}_i(L) = 0, \quad i = 1, 3, \\ \int_0^L (m \ddot{w} \tilde{y} + K_1 \ddot{w}_x \tilde{y}_x + K_2 w_{xx} \tilde{y}_{xx} - G_2 H \phi_x^2 \tilde{y}_x) dx \\ + k_1 \dot{w}_x(L) \tilde{y}_x(L) + k_2 \dot{w}(L) \tilde{y}(L) = 0. \end{aligned} \quad (14)$$

Now define the linear operators

$$\begin{aligned} \langle Ay, \psi \rangle_{\mathbb{V}' \times \mathbb{V}} &= (y, \psi)_{\mathbb{V} \times \mathbb{V}}, \quad \forall y, \psi \in \mathbb{V} \\ \langle B_0 \vec{y}, \vec{\psi} \rangle_{\mathbb{H}' \times \mathbb{H}} &= \begin{bmatrix} 0_{2 \times 1} \\ k_2 y_3(L) \psi_3(L) \end{bmatrix}, \quad \forall \vec{y}, \vec{\psi} \in \mathbb{H} \\ \langle D_0 \vec{y}, \vec{\psi} \rangle_{\mathbb{H}' \times \mathbb{H}} &= \begin{bmatrix} s_1 y_1(L) \psi_1(L) \\ s_3 y_2(L) \psi_2(L) \\ k_1 (y_3)_x(L) (\psi_3)_x(L) \end{bmatrix}, \quad \forall \vec{y}, \vec{\psi} \in \mathbb{V}. \end{aligned} \quad (15)$$

Let  $\mathcal{M} : H_L^1(0, L) \rightarrow (H_L^1(0, L))'$  be a linear operator defined by

$$\langle \mathcal{M} \psi, \tilde{\psi} \rangle_{(H_L^1(0,L))', H_L^1(0,L)} = \int_0^L (m \psi \tilde{\psi} + K_1 \psi_x \tilde{\psi}_x) dx. \quad (16)$$

From the Lax-Milgram theorem  $\mathcal{M}$  and  $A$  are canonical isomorphisms from  $H_L^1(0, L)$  onto  $(H_L^1(0, L))'$  and from  $\mathbb{V}$  onto  $\mathbb{V}'$ , respectively. Assume that  $Ay \in \mathbb{V}'$ , then we can formulate the variational equation above into the following form

$$M\ddot{y} + Ay + D_0\dot{y} + B_0\dot{y} = 0 \quad (17)$$

where  $M = [\rho_1 h_1 I \quad \rho_3 h_3 I \quad \mathcal{M}]$  is an isomorphism from  $\mathbb{H}$  onto  $\mathbb{H}'$ . Next we introduce the linear unbounded operator by

$$\mathcal{A} : \text{Dom}(\mathcal{A}) \times \mathbb{V} \subset \mathcal{H} \rightarrow \mathcal{H} \quad (18)$$

where  $\mathcal{A} = \begin{bmatrix} 0_{3 \times 3} & -I_{3 \times 3} \\ M^{-1}A & M^{-1}D_0 \end{bmatrix}$  with

$$\text{Dom}(\mathcal{A}) = \{(\vec{z}, \vec{\tilde{z}}) \in \mathbb{V} \times \mathbb{V}, A\vec{z} \in \mathbb{V}'\}.$$

and if  $\text{Dom}(\mathcal{A})'$  is the dual of  $\text{Dom}(\mathcal{A})$  pivoted with respect to  $\mathcal{H}$ , we define the control operators  $B$  and  $D$

$$B \in \mathcal{L}(\mathbb{C}, \text{Dom}(\mathcal{A})'), \text{ with } B = \begin{bmatrix} 0_{3 \times 1} \\ M^{-1}B_0 \end{bmatrix} \quad (19)$$

Writing  $\varphi = [v^1, v^3, w, \dot{v}^1, \dot{v}^3, \dot{w}]^T$ , the control system (11)-(12) with the feedback controllers can be put into the state-space form

$$\dot{\varphi} + \mathcal{A}\varphi + B\varphi = 0, \quad \varphi(x, 0) = \varphi^0. \quad (20)$$

**Lemma 4.1:** The operator  $\mathcal{A}$  defined by (18) is maximal monotone in the energy space  $\mathcal{H}$ , and  $\text{Range}(I + \mathcal{A}) = \mathcal{H}$ .

**Proof:** Let  $\vec{z} \in \text{Dom}(\mathcal{A})$ . A simple calculation using integration by parts and the boundary conditions yields

$$\begin{aligned} \left\langle \mathcal{A} \begin{bmatrix} \vec{z}_1 \\ \vec{z}_2 \end{bmatrix}, \begin{bmatrix} \vec{z}_1 \\ \vec{z}_2 \end{bmatrix} \right\rangle_{\mathcal{H} \times \mathcal{H}} &= \left\langle \begin{bmatrix} -\vec{z}_2 \\ M^{-1}A\vec{z}_1 \end{bmatrix}, \begin{bmatrix} \vec{z}_1 \\ \vec{z}_2 \end{bmatrix} \right\rangle_{\mathcal{H}} \\ &= \langle -\vec{z}_2, \vec{z}_1 \rangle_{\mathbb{V}' \times \mathbb{V}} + \langle M^{-1}(A\vec{z}_1 + D_0\vec{z}_2), \vec{z}_2 \rangle_{\mathbb{H} \times \mathbb{H}} \\ &= -\overline{\langle A\vec{z}_1, \vec{z}_2 \rangle_{\mathbb{V}' \times \mathbb{V}}} + \langle A\vec{z}_1 + D_0\vec{z}_2, \vec{z}_2 \rangle_{\mathbb{H}' \times \mathbb{H}}. \end{aligned}$$

Since  $\vec{z} = \begin{bmatrix} \vec{z}_1 \\ \vec{z}_2 \end{bmatrix} \in \text{Dom}(\mathcal{A})$ , then  $A\vec{z}_1 + D_0\vec{z}_2 \in \mathbb{V}'$  and  $\vec{z}_2 \in \mathbb{V}$  so that

$$\begin{aligned} \langle A\vec{z}_1 + D_0\vec{z}_2, \vec{z}_2 \rangle_{\mathbb{H}' \times \mathbb{H}} &= \langle A\vec{z}_1 + D_0\vec{z}_2, \vec{z}_2 \rangle_{\mathbb{V}' \times \mathbb{V}} \\ &= \langle A\vec{z}_1, \vec{z}_2 \rangle_{\mathbb{V}' \times \mathbb{V}} + \langle D_0\vec{z}_2, \vec{z}_2 \rangle_{\mathbb{V}' \times \mathbb{V}}. \end{aligned}$$

Hence  $\text{Re} \langle A\vec{z}, \vec{z} \rangle_{\mathcal{H} \times \mathcal{H}} = \langle D_0\vec{z}_2, \vec{z}_2 \rangle_{\mathbb{V}' \times \mathbb{V}} \geq 0$ . We next verify the range condition. Let  $\vec{z} = \begin{bmatrix} \vec{z}_1 \\ \vec{z}_2 \end{bmatrix} \in \mathcal{H}$ . We

prove that there exists a  $\vec{y} = \begin{bmatrix} \vec{y}_1 \\ \vec{y}_2 \end{bmatrix} \in \text{Dom}(\mathcal{A})$  such that

$(I + \mathcal{A})\vec{y} = \vec{z}$ . A simple computation shows that proving this is equivalent to proving  $\text{Range}(M + A + D_0) = \mathbb{H}'$ , i.e., for every  $\vec{f} \in \mathbb{H}'$  there exists a unique solution  $\vec{z} \in \mathbb{H}$  such that  $(M + A + D_0)\vec{z} = \vec{f}$ . This obviously follows from the Lax Milgram's theorem.  $\square$

**Proposition 4.1:** The operator  $B$  is a monotone compact operator on  $\mathbb{H}$ .

**Proof:** Let  $\begin{bmatrix} \vec{y} \\ \vec{z} \end{bmatrix} \in \mathbb{H}$ . Then  $\left\langle B \begin{bmatrix} \vec{y} \\ \vec{z} \end{bmatrix}, \begin{bmatrix} \vec{y} \\ \vec{z} \end{bmatrix} \right\rangle_{\mathcal{H}} = k_2 |z_3(L)|^2$ . The compactness follows from the fact that  $M^{-1}$  is a canonical isomorphism from  $\mathbb{H}$  to  $\mathbb{H}'$ , and the fact that  $B$  is a rank-one operator, hence compact from  $\mathbb{H}$  to  $\mathbb{H}'$ .  $\square$

### A. Description of $\text{Dom}(\mathcal{A})$

**Proposition 4.2:** Let  $\vec{u} = (\vec{y}, \vec{z})^T \in \mathcal{H}$ . Then  $\vec{u} \in \text{Dom}(\mathcal{A})$  if and only if the following conditions hold:

$$\begin{aligned} \vec{y} &\in (H^2(0, L) \cap H_L^1(0, L))^2 \times (H^3(0, L) \cap H_L^2(0, L)) \\ \vec{z} &\in V \text{ such that } (y_1)_x = (y_2)_x = (y_4)_{xx} \mid_{x=L} = 0. \end{aligned}$$

Moreover, the resolvent of  $\mathcal{A}$  is compact in the energy space  $\mathcal{H}$ .

**Proof:** Let  $\vec{u} = \begin{pmatrix} \vec{y} \\ \vec{z} \end{pmatrix} \in \mathcal{H}$  and  $\vec{u} = \begin{pmatrix} \vec{y} \\ \vec{z} \end{pmatrix} \in \text{Dom}(\mathcal{A})$  such that  $\mathcal{A}\vec{u} = \vec{u}$ . Then we have  $-\vec{z} = \vec{y} \in V$ ,  $A\vec{y} + D_0\vec{z} = M\vec{z}$ , and therefore,  $\langle \vec{y}, \vec{\varphi} \rangle_V = \langle \vec{z}, \vec{\varphi} \rangle_H$  for all  $\vec{\varphi} \in V$ . Let  $\vec{\psi} = [\psi_1, \psi_2, \psi_3]^T \in (C_0^\infty(0, L))^4$ . We define  $\varphi_i = \psi_i$  for  $i = 1, 2$ , and  $\varphi_3 = \int_0^x \psi_3(s)ds$ . Since  $\vec{\varphi} \in V$ , inserting  $\vec{\varphi}$  into the above equation yields

$$\begin{aligned} &\int_0^L \left\{ -\alpha_1^1 h_1 (y_1)_{xx} \bar{\psi}_1 - \alpha_1^3 h_1 (y_2)_{xx} \bar{\psi}_2 - K_2 (y_3)_{xxx} \bar{\psi}_3 \right. \\ &\quad \left. + \frac{G_2}{h_2} (-y_1 + y_2 + H(y_3)_x) (-\bar{\psi}_1 + \bar{\psi}_2 + H(\bar{\psi}_3)_x) \right\} dx \\ &\quad + s_1 (z_1)_x(L) (\psi_1)_x(L) + s_3 z_2(L) \psi_2(L) + k_1 (z_3)_x(L) \psi_3(L) \\ &= \int_0^L \left\{ \left( \int_1^x m \tilde{z}_4 ds + K_1 (\tilde{z}_4)_x \right) \bar{\psi}_4 + \rho_1 h_1 \tilde{z}_1 \bar{\psi}_1 \right. \\ &\quad \left. + \rho_3 h_3 \tilde{z}_2 \bar{\psi}_2 + \mu h_3 \tilde{z}_3 \bar{\psi}_3 \right\} dx \end{aligned}$$

for all  $\vec{\psi} \in (C_0^\infty(0, L))^3$ . Therefore it follows that  $\vec{y} \in (H^2(0, L) \cap H_L^1(0, L))^2 \times (H^3(0, L) \cap H_L^2(0, L))$ .

Next let  $\vec{\psi} \in \mathcal{H}$ . We define

$$\varphi_i = \int_0^x \psi_i(s)ds, \quad i = 1, \dots, 3. \quad (21)$$

Obviously  $\vec{\varphi} \in V$ . Then plugging (21) into (31) yields

$$\begin{aligned} 0 &= (\alpha_1^1 h_1 (y_1)_x(1) + s_1 z_1(L)) \bar{\psi}_1(1) + \alpha_1^3 h_3 (y_2)_x(1) \\ &\quad + s_3 z_2(L) \bar{\psi}_2(1) + (k_1 (y_3)_{xx}(L) + k_1 (z_3)_x(L)) (\bar{\psi}_3)_x(L) \end{aligned}$$

for all  $\psi \in \mathcal{H}$ . Hence,

$$\begin{aligned} \alpha_1^1 h_1 (y_1)_x(1) + s_1 z_1(L) &= \alpha_1^3 h_3 (y_2)_x(1) + s_3 z_2(L) = 0 \\ k_1 (y_3)_{xx}(L) + k_1 (z_3)_x(L) &= 0. \end{aligned}$$

Now let  $\vec{y} = \begin{bmatrix} \vec{y}_1 \\ \vec{y}_2 \end{bmatrix} \in \text{Dom}(\mathcal{A})$  and  $\vec{z} = \begin{bmatrix} \vec{z}_1 \\ \vec{z}_2 \end{bmatrix}$  such that  $(I + \mathcal{A})\vec{y} = \vec{z}$ . By Proposition 4.2 and Lemma 4.1, the compactness of the resolvent follows.  $\square$

**Lemma 4.2:** The eigenvalue problem

$$\begin{cases} \alpha_1^i h_i z_{xx}^i + \kappa(i) G_2 \phi^2 = \lambda^2 \rho_i h_i z^i, & i = 1, 3 \\ -K_2 u_{xxxx} + G_2 H \phi_x^2 = \lambda^2 (mu - K_1 u_{xx}), \end{cases} \quad (22)$$

with the overdetermined boundary conditions

$$\begin{aligned} u(0) = u_x(0) = z^i(0) = z^i(L) = z_x^i(L) = 0, & \quad i = 1, 3, \\ u(L) = u_x(L) = u_{xx}(L) = u_{xxx}(L) = 0 \end{aligned} \quad (23)$$

has only the trivial solution.

**Proof:** Now multiply the equations in (22) by  $x\bar{u}_x - 3\bar{u}$ ,  $x\bar{z}_x^1 - 2\bar{z}^1$ , and  $x\bar{z}_x^3 - 2\bar{z}^3$ , respectively, integrate by parts on  $(0, L)$ , and add them up:

$$\begin{aligned} 0 &= \int_0^L \left\{ -\alpha_1^1 h_1 |z_x^1|^2 - \alpha_1^3 h_3 |z_x^3|^2 - 3\rho_1 h_1 \lambda^2 |z^1|^2 \right. \\ &\quad - 3\rho_3 h_3 \lambda^2 |z^3|^2 - 4m\lambda^2 |u|^2 - 2K_1 \lambda^2 |u_x|^2 \\ &\quad - G_2 h_2 \bar{\phi}^2 (z\phi_x^2) - 3G_2 h_2 |\phi^2|^2 - K_2 \bar{u}_{xxxx} (xu_x) \\ &\quad + \alpha_1^1 h_1 \bar{z}_{xx}^1 (xz_x^1) + \alpha_1^3 h_3 \bar{z}_{xx}^3 (xz_x^3) - \rho_1 h_1 \lambda^2 \bar{z}^1 (xz_x^1) \\ &\quad \left. - \rho_3 h_3 \lambda^2 \bar{z}^3 (xz_x^3) - \lambda^2 (m\bar{u} - K_1 u_{xx}) (xu_x) \right\} dx \quad (24) \end{aligned}$$

Now consider the conjugate eigenvalue problem corresponding to (22)-(23). Now multiply the equations in the conjugate problem by  $xu_x + 2u$ ,  $xz_x^1 + 3z^1$ , and  $xz_x^3 + 3z^3$ , respectively, integrate by parts on  $(0, L)$ , and add them up:

$$\begin{aligned} 0 &= \int_0^L \left\{ 3\alpha_1^1 h_1 |z_x^1|^2 + 3\alpha_1^3 h_3 |z_x^3|^2 + 3\bar{\lambda}^2 \rho_1 h_1 |z^1|^2 \right. \\ &\quad + 3\rho_3 h_3 \bar{\lambda}^2 |z^3|^2 + 2m\bar{\lambda}^2 |u|^2 + 2K_1 \bar{\lambda}^2 |u_x|^2 + 2K_2 |u_{xx}|^2 \\ &\quad + G_2 h_2 \bar{\phi}^2 (z\phi_x^2) + 3G_2 h_2 |\phi^2|^2 + K_2 \bar{u}_{xxxx} (xu_x) \\ &\quad - \alpha_1^1 h_1 \bar{z}_{xx}^1 (xz_x^1) - \alpha_1^3 h_3 \bar{z}_{xx}^3 (xz_x^3) + \rho_1 h_1 \bar{\lambda}^2 \bar{z}^1 (xz_x^1) \\ &\quad \left. + \rho_3 h_3 \bar{\lambda}^2 \bar{z}^3 (xz_x^3) + \bar{\lambda}^2 (m\bar{u} - K_1 u_{xx}) (xu_x) \right\} dx \quad (25) \end{aligned}$$

Finally, adding (24) and (25), considering only the real part of the expression above and all eigenvalues are located on the imaginary axis, i.e.  $\lambda = \mp i\nu$ , yields

$$\int_0^L \left( K_2 |u_{xx}|^2 + m\nu^2 |u|^2 + \sum_{i=1,3} (\alpha_1^i h_i |z_x^i|^2) \right) dx = 0$$

This implies that  $u = z^1 = z^3 \equiv 0$  by (23). In the case of  $\lambda = 0$ , we have

$$\begin{cases} \alpha_1^i h_i z_{xx}^i + \kappa(i) G_2 \phi^2 = 0, & i = 1, 3, \\ -K_2 u_{xxxx} + G_2 H \phi_x^2 = 0. \end{cases} \quad (26)$$

We simplify the above equation to

$$\begin{cases} -\phi_{xx}^2 + \left( \frac{1}{\alpha_1^1 h_1} + \frac{1}{\alpha_1^3 h_3} \right) \phi^2 = -H u_{xxx} \\ -K_2 u_{xxxx} + G_2 H \phi_x^2 = 0, \end{cases} \quad (27)$$

Since the operator  $J = -D_x^2 + \left( \frac{1}{\alpha_1^1 h_1} + \frac{1}{\alpha_1^3 h_3} \right) I$  is a non-negative operator on  $H_*^2(0, L) = \{\psi \in H^2(0, L) : \psi(0) = \psi_x(L) = 0\}$ , we obtain

$$-K_2 u_{xxxx} - G_2 H^2 (J^{-1} u_{xxx})_x = 0,$$

and since  $K_2 D_x^4 + D_x J^{-1} D_x^3$  is a positive operator on its domain,  $u = \phi^2 = 0$ . And therefore,  $u = z^1 = z^3 \equiv 0$ .

**Remark 4.1:** (i) The result obtained above is not valid once we remove the boundary condition  $u(L) = 0$ .

(ii) The analogous result obtained in [12] was for either clamped, hinged, or mixed boundary conditions. It does require  $u(L) = 0$ . The interesting question is whether one can get the same result with only the three boundary conditions  $u_x(L) = u_{xx}(L) = u_{xxx}(L) = 0$  for the case  $k_1 \neq 0, k_2 \equiv 0$ .

**Theorem 4.1:** The semigroup generated by  $(\mathcal{A} + B)$  is strongly stable in  $\mathcal{H}$ .

**Proof:** We know that the system is dissipative, i.e.  $\left\langle (\mathcal{A} + B) \begin{bmatrix} \tilde{z} \\ \tilde{z} \end{bmatrix}, \begin{bmatrix} \tilde{z} \\ \tilde{z} \end{bmatrix} \right\rangle_{\mathcal{H}} \leq 0$ . This result together with Lemma 4.2 imply that there are no eigenvalues on the imaginary axis. The conclusion follows.  $\square$

Now we consider the decomposition  $\mathcal{A} + B = (\mathcal{A}_d + B) + \mathcal{A}_\phi$  of the semigroup generator of the original problem (18) where  $\mathcal{A}_d + B$  is the semigroup generator of the decoupled system, i.e.  $\phi^2 \equiv 0$  in (11)-(12),

$$\begin{cases} \rho_i h_i \ddot{v}^i - \alpha_1^i h_i v_{xx}^i = 0, & i = 1, 3, \\ m \ddot{w} - K_1 \ddot{w}_{xx} + K_2 w_{xxxx} = 0, \end{cases} \quad (28)$$

with the boundary and initial conditions

$$\begin{aligned} v^i(0) &= w(0) = w_x(0) = 0, \\ \alpha_1^i h_i v_x^i(L) &= -s_i \gamma^i \dot{v}^i(L), \quad i = 1, 3, \\ K_2 w_{xx}(L) &= -k_1 \dot{w}_x(L) \\ K_1 \ddot{w}_x(L) - K_2 w_{xxx}(L) &= k_2 \dot{w}(L) \\ (v^1, v^3, w, \dot{v}^1, \dot{v}^3, \dot{w})(x, 0) &= (v_0^1, v_0^3, w_0, v_1^1, v_1^3, w_1). \end{aligned} \quad (29)$$

The operator  $\mathcal{A}_\phi : \mathcal{H} \rightarrow \mathcal{H}$  is the coupling between the layers defined as the following

$$\mathcal{A}_\phi \mathbf{y} = \begin{pmatrix} 0_{3 \times 1} \\ \mathcal{M}^{-1} \begin{pmatrix} H G_2 & \phi_x^2 \\ \frac{G_2}{h_1 \rho_1} \phi^2 \\ -\frac{G_2}{h_3 \rho_3} \phi^2 \end{pmatrix} \end{pmatrix}. \quad (30)$$

where  $\mathbf{y} = (w, u^1, u^3, \tilde{w}, \tilde{v}^1, \tilde{v}^3)$  and  $\phi^2 = \frac{1}{h_2} (-u^1 + u^3 + H u_x)$ . Let  $E_d(t)$  be natural energy corresponding to the system (28)-(29), i.e.  $\phi^2 \equiv 0$  in (13).

**Theorem 4.2:** Let  $\mathcal{A}_d + B$  be the infinitesimal generator of the semigroup corresponding to the solutions of (28)-(12). Then the semigroup  $\{e^{(\mathcal{A}_d + B)t}\}_{t \geq 0}$  is exponentially stable in  $\mathcal{H}$ .

**Proof:** Note that the equations in (28) are completely decoupled. The exponential stability of the semigroup  $e^{(\mathcal{A}_d + B)t}$  follows from the exponential stability of wave equations [12] and the Rayleigh beam equation [14].

**Lemma 4.3:** The operator  $A_\phi : \mathcal{H} \rightarrow \mathcal{H}$  defined in (30) is compact.

When  $(w, u^1, u^3, \tilde{w}, \tilde{u}^1, \tilde{u}^3) \in \mathcal{H}$ , we have  $w \in H_L^2(0, L)$  and  $u^1, u^3 \in (H_L^1(0, L))^2$ , and therefore  $\phi^2 \in H_L^1(0, L)$ . Since  $\mathcal{M} : H_L^2(0, L) \rightarrow L^2(0, L)$  remains an isomorphism, the last terms in (30) satisfy

$$\mathcal{M}^{-1}(\phi_x^2) \in H_L^2(0, L), \quad \phi^2 \in H_L^1(0, L), \quad (31)$$

and  $H_L^2(0, L) \times H_L^1(0, L)$  is compactly embedded in  $H_L^1(0, L) \times (L^2(0, L))^2$ . Hence the operator  $A_\phi$  is compact in  $\mathcal{H}$ .

**Theorem 4.3:** Then the semigroup  $\{e^{(\mathcal{A} + B)t}\}_{t \geq 0}$  is exponentially stable in  $\mathcal{H}$ .

**Proof:** The semigroup  $\mathcal{A} + B = \mathcal{A}_d + B + \mathcal{A}_\phi$  is strongly stable on  $\mathcal{H}$  by Theorem 4.1, and the operator  $\mathcal{A}_\phi$  is a compact in  $\mathcal{H}$  by Lemma 4.3. Therefore, since the semigroup generated by  $(\mathcal{A}_d + B + \mathcal{A}_\phi) - \mathcal{A}_\phi$  is uniformly exponentially stable in  $\mathcal{H}$  then the semigroup

$\mathcal{A} = (\mathcal{A}_d + B + \mathcal{A}_\phi)$  is uniformly exponentially stable in  $\mathcal{H}$  by e.g., the perturbation theorem of [18].

## V. FUTURE RESEARCH

A relevant research problem is whether we can recover the polynomial stability for certain combinations of material properties and for more regular initial data in (9)-(10). For a single piezoelectric beam, this question is answered in [11].

## REFERENCES

- [1] A. Baz, Boundary Control of Beams Using Active Constrained Layer Damping, *J. Vib. Acoust.* **119-2** (1997), 166-172.
- [2] C.Y.K. Chee, L. Tong, and G.P. Steven, A review on the modelling of piezoelectric sensors and actuators incorporated in intelligent structures, *J. Intell. Mater. Syst. Struct.* **9** (1998), 3-19.
- [3] R.A. DiTaranto, Theory of vibratory bending for elastic and viscoelastic layered finitlength beams, *J. Appl. Mech.* **32** (1965), 881-886.
- [4] S.W. Hansen, Several Related Models for Multilayer Sandwich Plates, *Mathematical Models & Methods in Applied Sciences* **14-8** (2004), 1103-1132.
- [5] S.W. Hansen, A.Ö. Özer, Exact boundary controllability of an abstract Mead-Marcus Sandwich beam model, *The Proceedings of 49<sup>th</sup> IEEE Conf. on Decision & Control*, Atlanta, USA (2010), 2578-2583.
- [6] J.E. Lagnese, J.-L. Lions, *Modeling Analysis and Control of Thin Plates*, (Masson, Paris, 1988).
- [7] M.J. Lam, D. Inman, W. R. Saunders, Vibration Control through Passive Constrained Layer Damping and Active Control, *Journal of Intelligent Material Systems and Structures* (8-8) (1997), pp. 663-677.
- [8] D.J. Mead and S. Markus, The forced vibration of a three-layer, damped sandwich beam with arbitrary boundary conditions, *J. Sound Vibr.* **10** (1969), 163-175.
- [9] S. Miller and J.Jr. Hubbard, Observability of a Bernoulli - Euler Beam using PVF<sub>2</sub> as a Distributive Sensor, *The Seventh Conference on Dynamics & Control of Large Structures*, VPI & SU, Blacksburg, VA (1987), 375-390.
- [10] K.A. Morris, A.Ö. Özer, Modeling and stabilizability of voltage-actuated piezoelectric beams with magnetic effects, *SIAM J. Control Optim.* **52-4** (2014), 2371-2398.
- [11] A.Ö. Özer, Further stabilization and exact observability results for voltage-actuated piezoelectric beams with magnetic effects, *Mathematics of Control, Signals, and Systems* **27-2** (2015), 219-244.
- [12] A.Ö. Özer, and S.W. Hansen, Uniform stabilization of a multi-layer Rao-Nakra sandwich beam, *Evolution Equations and Control Theory* **2-4** (2013), 195-210.
- [13] Y.V.K.S. Rao and B.C. Nakra, Vibrations of unsymmetrical sandwich beams and plates with viscoelastic cores, *J. Sound Vibr.* **34-3** (1974), 309-326.
- [14] B. Rao, A compact perturbation method for the boundary stabilization of the Rayleigh beam equation, *Appl. Math. Optim.*, 3-33, pp. 253-264, 1996.
- [15] I.Y. Shen, A variational formulation, a work-energy relation and damping mechanisms of active constrained layer treatments, *Journal of Vibration and Acoustics* **119-2** (1997), 192-199.
- [16] R.C. Smith, *Smart Material Systems*, (Society for Industrial and Applied Mathematics, 2005).
- [17] R. Stanway, J.A. Rongong, N.D. Sims, Active constrained-layer damping: A state-of-the-art review, *Automation & Control Systems* **217-6** (2003), pp. 437-456.
- [18] R. Triggiani, Lack of uniform stabilization for noncontractive semigroups under compact perturbation, *Proc. Amer. Math. Soc.* **(105)** (1989), pp. 375-383.
- [19] M. Trindade and A. Benjendou, Hybrid Active-Passive Damping Treatments Using Viscoelastic and Piezoelectric Materials: Review and Assessment, *Journal of Vibration and Control* **8-6** (2002), pp. 699-745.